Hedging Cryptos with futures

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**Digital assets are here to stay**

- Markets for cryptocurrencies are maturing
  - Institutional investors are buying into it
  - Regulators are working hard to make stablecoins “safe“ (e.g. resolve issues of jurisdiction, financial stability)
  - Exchanges (e.g. CME) are issuing futures and options

**We are in the middle of a rapid decentralisation of financial markets!**
Digital assets are here to stay

As Bitcoin Rises, Institutions Make Crypto Market Impact

Barriers fall away but hedging remains a challenge; regulatory clarity will help

Friday, February 26, 2021

By John Hintze

https://www.garp.org/#!/risk-intelligence/market/investment-management/a1Z1W000005kZDGUA2
Motivation

Beer for Bitcoin, 2011

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Motivation

**Bitcoin futures**

- CME launched BTC Futures in December 2017 and options on futures in January 2020
- **Bitcoin Future**
  - Underlying: BTC Reference Rate (BRR), based on relevant bitcoin transaction on certain exchanges
  - Maturities: nearest two Decembers and nearest six consecutive months
  - Settlement: cash
- [https://www.cmegroup.com/trading/equity-index/us-index/bitcoin.html](https://www.cmegroup.com/trading/equity-index/us-index/bitcoin.html)
Hedging cryptos

- Hedging Bitcoin exposure with Bitcoin futures
  - Basis risk
  - BRR not traded
  - Ability of futures to hedge tail risks?

- Hedge other cryptos with Bitcoin?
  - High correlation,
  - Tail risks, extreme events?

- Two directions
  - Copulae
  - Risk measures

Source: skew.com, December 2019
Outline

- Motivation ✓
- Copula-based hedging
- Data
- Results
Hedging spot with futures

- Hedge portfolio return: $R_h^t = R_s^t - h R_f^t$,
  - $R_s^t$: spot return
  - $R_f^t$: futures return
  - $h$: hedge ratio

- Hedge portfolio return: $R_h^t = R_s^t - h R_f^t$

- Goal: Find optimal hedge ratio $h^*$

- Minimum-variance hedge ratio, e.g., variance as risk measure and elliptical return distribution

- Extensions: risk measures, copulae,

Ederington (1979), Harris and Shen (2006), Barbi and Romagnoli (2014)
Copulae

A (bivariate) copula is a distribution function on $[0,1]^2$ with standard uniform marginals, i.e. $C : [0,1]^2 \mapsto [0,1]$.

- Copulae differ only through the dependence between the marginals.
- Hoeffding’s Theorem captures that copulae allow to separate
  - modelling of the marginals
  - modelling of the dependence structure
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Hoeffding’s Theorem

Let $F_{X,Y}$ be a joint distribution function with margins $F_X$ and $F_Y$. Then there exists a copula $C$ such that for all $x, y \in \mathbb{R}$

$$F_{X,Y}(x, y) = C\{F_X(x), F_Y(y)\}$$

Wassili Hoeffding (1940)

An Introduction to Copulas

Nelsen (2005)

Copula Theory

Jaworski et al (2009)
Some copulae

All copulae are calibrated to a Spearman’s Rho of 0.75.
Examples of Copulae

Copula is nothing more than parameterising the dependency structure by a function, e.g.

Gaussian Copula

\[ C(u_1, u_2) = \Phi_\rho \left\{ \Phi^{-1}(u_1), \Phi^{-1}(u_2) \right\} \]

\[ = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp \left\{ -\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)} \right\} \, dx \, dy, \]

Gumbel

\[ C(u_1, u_2) = \exp \left[ -\left\{ (-\log u_1)^\theta + (-\log u_2)^\theta \right\}^{\frac{1}{\theta}} \right], \]

where \( u_1 \) and \( u_2 \) are \( F_X(X) \) and \( F_Y(Y) \) respectively.
Distribution of the hedge

Let \( X \) and \( Y \) be two rv's with corresponding abs cts copula \( C \) and marginals \( F_X \) and \( F_Y \).

Then, the distribution of \( Z = X - hY \) is given by

\[
F_Z(x) = 1 - \int_0^1 D_1 C \left[ u, F_Y \left\{ \frac{F_X^{-1}(u) - x}{h} \right\} \right] du
\]

Barbi and Romagnoli (2014)

Easy to show Hoeffding (1940), McNeil et al. (2005)

\[
D_1 C\{F_X(x), F_Y(y)\} = \frac{\partial}{\partial u} C(u, v) = P(Y \leq y | X = x)
\]
normierte Summenfunktion lautet folglich

\[ S_2(x, y) = \begin{cases} 0 & \text{für } y \leq -x, \\ x + y & \text{für } y \geq -x, \end{cases} \]

und entspricht der Fläche \((a, f; b, e, f)\).

Ist umgekehrt die normierte Summenfunktion durch (3.9) gegeben, so daß die Summenfläche mit der oberen Tetraederbegrenzung übereinstimmt, so betrachten wir die Kurve \(K\) der \(x, \eta\)-Ebene \(U(\eta) = V(\eta)\) ist. Da nach Voraussetzung \(s(\eta)\) im Intervall \((a, b)\) und \((\gamma, \delta)\) von Null verschieden ist, so ist die Kurve \(K\) im Raum \((a, \beta; \gamma, \delta)\) eine monoton zunehmende Kurve in den Punkten \((\xi, \eta)\) mit \(V(\eta) \geq U(\xi)\), also "oberhalb" der Kurve \(K\) gleich Null entsprechend ist in allen Punkten "unterhalb" von \(K\) gleich Null. Daraus folgt, daß für ein Wertepaar \((\xi, \eta)\) "links oben" von Punkt \(\xi, \eta\) für alle Punkte \((\xi, \eta)\) "oberhalb" der Kurve \(K\) gleich Null entsprechend ist in allen Punkten "unterhalb" von \(K\) gleich Null. Dies ist bedeutend damit, daß nur auf der monoton zunehmenden Kurve eine von Null verschiedene Wahrscheinlichkeit liegen kann. \(\eta\) ist folglich eine unendlich eindeutige Funktion von \(\xi\), und zwar nehmen \(\xi\) und \(\eta\) gleichzeitig zu.

Ganz analog sieht man, daß die normierte Summenfunktion (3.12), die durch die untere Tetraederbegrenzung dargestellt wird, dem Fall der monoton abnehmenden funktionellen Abhängigkeit entspricht. Ebenso wie es nur eine "unabhängige" normierte Verteilung gibt, gibt es also nur je eine normierte Verteilung, die im besten Grenzfall der unabhängigen eindeutigen funktionellen Abhängigkeit entspricht.

Wir haben festgestellt, daß die Summenfläche \(S(x, y)\) innerhalb des in Abb. 2 dargestellten Tetraeders verläuft. Weitere Aussagen über die Gestalt der Summenfläche ergeben man, wenn man die Ableitungen von \(S(x, y)\) nach \(x\) und \(y\) betrachtet. Aus (3.1) entnimmt man zunächst, daß

\[ \frac{\partial S(x, y)}{\partial x} = 2 \int_{-\frac{1}{2}}^{y} s(x, y') \, dy' \]

ist. Als Funktion von \(y\) steigt \(\frac{\partial S(x, y)}{\partial x}\) monoton von 0 bis 1, so daß

\[ 0 \leq \frac{\partial S(x, y)}{\partial x} \leq 1 \]

ist. Entsprechend ist

\[ 0 \leq \frac{\partial S(x, y)}{\partial y} \leq 1. \]
Risk measures

Variance: $\text{Var}(Z)$

$\text{Var}(Z)$

Value-at-risk (VaR)

$\text{VaR}_\alpha = -F_Z^{-1}(1 - \alpha)$

Expected Shortfall (ES)

$\text{ES}_\alpha = -\frac{1}{1 - \alpha} \int_0^{1-\alpha} F_Z^{-1}(p) dp$

Figure 4: Expectile and quantile loss functions at $\alpha = 0.01$ (left) and $\alpha = 0.50$ (right)
Spectral Risk Measures


\[ \rho_\phi = -\int_0^1 \phi(p) F_Z^{-1}(p) \, dp, \]

\( \phi(s), s \in [0,1], \) is the risk aversion function, a weighting function such that

(i) \( \phi \geq 0 \)

(ii) \( \int_0^1 \phi(p) \, dp = 1 \)

(iii) \( \phi' \leq 0 \)

With (iii) SRM’s are coherent risk measures.

Value-at-risk (VaR): \( \text{VaR}_\alpha = -F_Z^{-1}(1 - \alpha) \) with a delta fct \( \phi(p) \)
Spectral Risk Measures

Expected Shortfall (ES):\[ \text{ES}_\alpha = -\frac{1}{1 - \alpha} \int_0^{1-\alpha} F_Z^{-1}(p) \, dp \]

Exponential spectral risk measures: weighting function\[ \phi(p) = \lambda e^{-k(1-p)}, \quad \lambda \text{ a normalising constant, derived from exp utility fct:} \]
\[ \rho_\phi = \int_0^1 \phi(p) F_Z^{-1}(p) \, dp = \frac{k}{1 - e^{-k}} \int_0^1 e^{-k(1-p)} F_Z^{-1}(p) \, dp, \]

e.g. „spectral 10“ uses \( k=10 \)
Optimal hedge ratio

- Hedge portfolio return: $R_t^h = R_t^S - h R_t^F$, with $h$ hedge ratio
- Optimal hedge ratio:

$$h^* = \arg\min_h \rho(h)$$

where $\rho(h)$ is the risk of the hedge portfolio with hedge ratio $h$. 
BTC and its „future“

- Daily log returns, 23pm CET
- 29 May 2018 through 3 Feb 2021
- Spot: Coingecko Bitcoin /USD
- Future: CME BTC Future
- Sources: Bloomberg, Coingecko
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Time series

Log return: $r = \log(\frac{P_t}{P_{t-1}})$

Red dots are 10% extremes (upper and lower)

Naive Hedge: Spot - Future, i.e. $h = 1$
Time series

Cryptocurrency Regulatory Risk Index (CRRIX)

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Distribution

- Student $t$ distribution: $\nu = 7.95$
- Generalised Pareto distribution (EVT): tail index $1/\xi = 4.92$

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Spot and Future

![Data](image-url)
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Spot and Future empirical copula

Data
Calibration

Copulae

- Method of moments for copulae (Genest and Rivest, 1993; Oh and Patton, 2013)
- „Moments“:
  - Spearman’s Rho
  - Quantile dependence at 0.05, 0.1, 0.9, 0.95 quantiles
- Margins distribution: pdf by kernel density estimator (Gaussian kernel)

Optimal Hedge Ratio

- Draw samples from copulae, calculate risk measure w.r.t. $h$
- Find $h$ which minimise risk measure
- By Nelder-Mead simplex method by Python package Scipy
A First Glance to the Results

- Out-of-sample optimal $h$ from 2019 Sept to 2021 Mar
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Results

P&L

- Daily return from hedge, out-of-sample
- Recalibration every 5 days
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Results

P&L from static hedge, out-of-sample, 100 days, rolling every 5 days
P&L from static hedge, out-of-sample, rolling every 5 days
From left to right: Variance, VaR 99%, VaR 95%, ES 99%, ES 95%, Spectral 10
Interactive Reports

- https://francisliu2.github.io/francis.github.io/Copula name_risk measure.html
  - e.g. https://francisliu2.github.io/francis.github.io/Clayton_ERM k=10.html

- Available **copula names**:
  - Gaussian, t_Copula Frank, Clayton, Gumbel, Plackett, NIG_factor

- Available **risk measure names**:
  - ERM k=10, ES q=0.01, ES q=0.05, VaR q=0.01, VaR q=0.05, Variance
Quality of Hedge

- Hedging Effectiveness (Ederington, 1979)
- Mean square difference
- Downside Semi Variance
- Robustness: sensitivity of the procedure w.r.t. „outliers“
Hedge Effectiveness

- Hedge effectiveness (Ederington, 1979) captures percentage reduction in risk:

\[ 1 - \frac{\rho(r^h)}{\rho(r^S)} . \]

- Compare the hedging effectiveness among copulae under different risk measures.

- Measure the risk reduction by loss function, e.g. for the pair of Gaussian and Expected Shortfall 99%, we measure the HE:

\[ 1 - \frac{\text{ES } 99\%(r^h)}{\text{ES } 99\%(r^S)} . \]
Hedge Effectiveness

Out-of-Sample Hedging Effectiveness of Variance

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Hedge Effectiveness

Out-of-Sample Hedging Effectiveness of VaR 99%

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Hedge Effectiveness

Out-of-Sample Hedging Effectiveness of VaR 95%

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Hedge Effectiveness

Out-of-Sample Hedging Effectiveness of ES 99%

Gaussian, t_Copula, Clayton, Frank, Gumbel, Plackett, Gauss Mix Indep, NIG_factor, HR 1
Hedge Effectiveness

Out-of-Sample Hedging Effectiveness of ES 95%

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Hedge Effectiveness

Out-of-Sample Hedging Effectiveness of ERM k=10

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Root Mean Square Error

- Recall our final goal: to form a portfolio with spot and future such that the P&L is zero.
- A perfect hedge $r_h^{\text{ideal}} = 0$.
- Root Mean Square Error is intuitive measure to assess the quality of hedge.
- For portfolio return:
  \[ \text{RMSE}(r^h) = \sqrt{(r^h - 0)^2} \]
- Simply the standard deviation.
Root Mean Square Error

- Out-of-sample RMSE
- Rule out Frank Copula
- VaR 99% and ES 99% are slightly underperforming
- Observation: Minimising Variance in-sample data does not necessary minimise variance out-of-sample
- Reflects the highly unstable nature of the correlation between spot and future
Semi Variance

- Lower semi variance
  \[ \text{E} \left[ (X - \text{E}(X))^2 \cdot 1\{X \leq \text{E}(X)\} \right]^{\frac{1}{2}} \]

- See also McNeil (2005) and Markowitz (1991)

- Rule out Frank, VaR 99%, and ES 99%

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Robustness

- Intense discussions about „what is robust?“ Hampel vs. Huber
- Our case: jumps and correlation distress
- Market dynamics always exist, but do we want the optimal hedge ratio to react to extreme market changes?
  - Elon Musk tweets
  - A sudden large order from institutional investor
  - An incident of system failure in crypto exchanges
- Hampel’s infinitesimal approach: influence function
  \[ IF = \frac{\hat{h}_\rho(X_1, \ldots, X_n, z) - \hat{h}_\rho(X_1, \ldots, X_n)}{\frac{1}{n + 1}} \]
  - Artificial shock
  - Procedure of getting \( h \)
  - Read as the change of optimal hedge ratio if we add one shock (two dimensional, \( r^s \) and \( r^f \)) to the training data
Robustness

- t-Copula with different risk measures
- 300 data points from Dec 2018 to Feb 2020
- ES 99% and VaR 99% are very sensitive to outliers

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Conclusion

- Hedging with different copulae and risk measure produces mixed results:
  - Frank copula underperforms consistently in hedging effectiveness and robustness
  - NIG and Gaussian Mix produce small hedge ratios pre-Covid-19 pandemic
  - NIG factor produces good hedge effectiveness
  - Gumbel produces good results in P&L

- Risk measures as loss function
  - Min variance in-sample does not necessary min that of out-of-sample data
  - ES 99% and VaR 99% are sensitive to outliers

- Next step: Hedge other cryptos (e.g. CRIX index) with BTC futures
Blockchain Research Center

1. The BRC data pool

Cryptocurrency - Index - Data - Derivatives
VCRIX - Volatility Index
financialriskmeter
Quantlet

2. Joint BRCs

blockchain-research-center.de

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References


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